

A COMBINATORIAL CLASSIFICATION OF POSTCRITICALLY FIXED NEWTON MAPS

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ABSTRACT. We give a combinatorial classification for the class of postcritically fixed Newton maps of polynomials and indicate potential for extensions.

As our main tool, we show that for a large class of Newton maps that includes all hyperbolic ones, every component of the basin of an attracting fixed point can be connected to ∞ through a finite chain of such components.

1. INTRODUCTION

One of the most important open problems in rational dynamics is understanding the structure of the space of rational functions of a fixed degree $d \geq 2$. This problem is today wide open.

Aside from being a useful tool for numerical root-finding, Newton maps of polynomials form an interesting subset of the space of rational maps that is more accessible for studying than the full space of rational maps. Hence, a partial goal in the classification of *all* rational maps can be to gain an understanding of the space of Newton maps.

In this paper, we present a theorem that structures the dynamical plane of postcritically finite Newton maps, and then use this result to construct a graph that classifies those Newton maps whose critical orbits all terminate at fixed points. Newton maps of degree 1 and 2 are trivial, and we exclude these cases from our investigation. Let us make precise what we mean by a Newton map.

Definition 1.1 (Newton Map). *A rational function $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree $d \geq 3$ is called a Newton map if ∞ is a repelling fixed point of f and for each fixed point $\xi \in \mathbb{C}$, there exists an integer $m \geq 1$ such that $f'(\xi) = (m-1)/m$.*

This definition is motivated by the following observation, which is a special case of [RS, Proposition 2.8] (the case of superattracting fixed points, i.e. every $m = 1$, goes back to [He, Proposition 2.1.2]).

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Proposition 1.2 (Head's Theorem). *A rational map f of degree $d \geq 3$ is a Newton map if and only if there exists a polynomial $p : \mathbb{C} \rightarrow \mathbb{C}$ such that for $z \in \mathbb{C}$, $f(z) = z - p(z)/p'(z)$. \square*

Observe that f and p have the same degree if and only if p has d distinct roots. Note also that some rational maps arise as Newton maps of entire functions, for example for $h = pe^q$ where p, q are complex polynomials, the map N_h is rational.

Definition 1.3 (Immediate Basin). *Let f be a Newton map and $\xi \in \mathbb{C}$ a fixed point of f . Let $B_\xi = \{z \in \mathbb{C} : \lim_{n \rightarrow \infty} f^{on}(z) = \xi\}$ be the basin (of attraction) of ξ . The component of B_ξ containing ξ is called the immediate basin of ξ and denoted U_ξ .*

Clearly, B_ξ is open and by a theorem of Przytycki [Pr], U_ξ is simply connected and $\infty \in \partial U_\xi$ is an accessible boundary point (in fact, a result of Shishikura [S] implies that every component of the Fatou set is simply connected).

Our first result is the following.

Theorem 1.4 (Preimages Connected). *Let $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a Newton map with attracting fixed points $\xi_1, \dots, \xi_d \in \mathbb{C}$, and let U'_0 be a component of some B_{ξ_i} . Then, U'_0 can be connected to ∞ by the closures of finitely many components U'_1, \dots, U'_k of $\bigcup_{i=1}^d B_{\xi_i}$.*

More precisely, there exists a curve $\gamma : [0, 1] \rightarrow \widehat{\mathbb{C}}$ such that $\gamma(0) = \infty$, $\gamma(1) \in U'_0$ and for every $t \in [0, 1]$, there exists $m \in \{0, 1, \dots, k\}$ such that $\gamma(t) \in \overline{U'_m}$.

We will see that γ can be chosen to consist of the closures of internal rays in the U'_m . Theorem 1.4 allows to describe how the components of the basins are connected to each other. Thus, it is a basis for a combinatorial classification of certain Newton maps: Theorems 1.5 and 1.6 show that the combinatorics of these connections suffice to describe postcritically fixed Newton maps uniquely.

We call a Newton map *postcritically fixed* if all its critical points are mapped onto fixed points after finitely many iterations. If f is a postcritically fixed Newton map, Theorem 1.4 allows to structure the entire Fatou set, because each Fatou component is in the basin of some attracting fixed point. Then, we construct the channel diagram Δ of f , which is the union of the accesses from finite fixed points of f to ∞ (see Section 2). Denote by Δ_n the connected component of $f^{-n}(\Delta)$ containing ∞ . Pulling back Δ several times we get a connected graph Δ_n that contains the forward orbits of all critical points, similar to the Hubbard tree of a postcritically finite polynomial.

We introduce the notion of an *abstract Newton graph*, which is a pair (g, Γ) of a map g acting on a graph Γ that satisfies certain conditions, (see Definition 4.6). In particular, the conditions on (g, Γ) allow g to be extended to the branched covering \bar{g} of the whole sphere \mathbb{S}^2 .

We show that for every *abstract Newton graph* there exists a unique up to affine conjugacy postcritically fixed Newton map realizing it.

The assignments of a Newton map to an abstract Newton graph and vice versa are injective and inverse to each other, so we give a combinatorial classification of postcritically fixed Newton maps by way of abstract Newton graphs. Thus, our main results are the following (see Sections 4 and 5 for the precise definitions).

Theorem 1.5 (Newton Map Generates Newton Graph). *Every postcritically fixed Newton map f gives rise to a unique up to equivalence relation abstract Newton graph. More precisely, there exists a unique $N \in \mathbb{N}$ such that (Δ_N, f) is an abstract Newton graph.*

If f_1 and f_2 are Newton maps with channel diagrams Δ_1 and Δ_2 such that $(\Delta_{1,N}, f_1)$ and $(\Delta_{2,N}, f_2)$ are equivalent as abstract Newton graphs, then f_1 and f_2 are affinely conjugate.

Theorem 1.6 (Newton Graph Generates Newton Map). *Every abstract Newton graph is realized by a postcritically fixed Newton map which is unique up to affine conjugacy. More precisely, let (Γ, g) be an abstract Newton graph. Then, there exists a postcritically fixed Newton map f with channel diagram $\hat{\Delta}$ such that (\bar{g}, Γ') and $(f, \hat{\Delta}'_{N_\Gamma})$ are Thurston equivalent as marked branched coverings.*

Moreover, if f realizes two abstract Newton graphs (Γ_1, g_1) and (Γ_2, g_2) , then the two abstract Newton graphs are equivalent.

Our construction of an abstract Newton graph can be done for all postcritically finite Newton maps, but will in general not contain the orbits of *all* critical points, and thus not describe the combinatorics of the entire Fatou set (note that there are rational maps with *buried* Fatou components that are not attached to any other Fatou component (it is not hard to find Newton maps with this property either). An extreme example of this behavior is provided by rational maps with Sierpinski Julia sets, see e.g. [Mi1, Appendix F]). It seems likely however that with Theorem 1.4 and additional combinatorial objects that describe any strictly periodic or preperiodic critical points, a classification of at least all hyperbolic Newton maps can be achieved. Thus, our results are a first step towards a combinatorial classification of Newton maps, and in particular of all hyperbolic components in the space of Newton maps. They may also be a basis for transporting the powerful concept of Yoccoz puzzles, which has been used to prove local connectivity of the Julia set for many classes of polynomials, to the setting of Newton maps beyond the cubic case (Roesch has successfully applied Yoccoz puzzles to cubic Newton maps [Ro]).

A number of people have studied Newton maps and used combinatorial models to structure the parameter spaces of some Newton maps. Janet Head [He] introduced the *Newton tree* to characterize postcritically finite cubic Newton maps. Tan Lei [TL] built upon this work and gave a classification of postcritically finite cubic Newton maps in terms of matings and captures.

Jiaqi Luo [Lu] extended some of these results to “unicritical” Newton maps, i.e. Newton maps of arbitrary degree with only one *free* (non-fixed) critical value. The present work can be seen as an extension of these results beyond the setting of a single free critical value. The main differences to this setting are that the channel diagram is in general not a tree anymore and that in the presence of more than one non-fixed critical value, the iterated preimages of the channel diagram may be disconnected.

This article is structured as follows. In Section 2, we introduce the concept of a channel diagram for Newton maps and discuss some of its properties. We use the channel diagram and its preimages to prove Theorem 1.4 in Section 3. In Section 4, we introduce abstract Newton graphs and prove Theorem 1.5. Theorem 1.6 is proved in Section 5, following a review of some aspects of Thurston theory. We also give an introduction to the combinatorics of *arc systems* and state a result by Kevin Pilgrim and Tan Lei that restricts the possibilities of how arc systems and Thurston obstructions can intersect.

1.1. Notation. Let f be a Newton map. A point $z \in \mathbb{C}$ is called a *pole* if $f(z) = \infty$ and a *prepole* if $f^{\circ k}(z) = \infty$ for some minimal $k > 1$. If $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a branched covering map, we call a point $z \in \widehat{\mathbb{C}}$ a *critical point* if g is not injective in any neighborhood of z . For the Newton map f , this is equivalent to saying that $z \in \mathbb{C}$ and $f'(z) = 0$, because ∞ is never a critical point of f . It follows from the Riemann-Hurwitz formula [Mi2, Theorem 7.2] that a degree- d branched covering map of $\widehat{\mathbb{C}}$ has exactly $2d - 2$ critical points, counting multiplicities.

Definition 1.7 (Postcritically Fixed). *Let $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be a branched covering map of degree $d \geq 2$ with (not necessarily distinct) critical points c_1, \dots, c_{2d-2} . We denote the set of critical values of g by*

$$\text{CV}(g) := \{g(c_1), \dots, g(c_{2d-2})\}.$$

Then, g is called postcritically finite if the set

$$P_g := \bigcup_{n \geq 0} g^{\circ n}(\text{CV}(g))$$

is finite. We say that g is postcritically fixed if there exists $N \in \mathbb{N}$ such that for each $i \in \{1, \dots, 2d - 2\}$, $g^{\circ N}(c_i)$ is a fixed point of g .

Definition 1.8 (Access to ∞). *Let U_ξ be the immediate basin of the fixed point $\xi \in \mathbb{C}$. Consider a curve $\Gamma : [0, \infty) \rightarrow U_\xi$ with $\Gamma(0) = \xi$ and $\lim_{t \rightarrow \infty} \Gamma(t) = \infty$. Its homotopy class within U_ξ defines an access to ∞ for U_ξ , in other words a curve Γ' with the same properties lies in the same access as Γ if the two curves are homotopic in U_ξ with the endpoints fixed.*

Proposition 1.9 (Accesses). (c.f. [HSS]) *Let f be a Newton map of degree $d \geq 3$ and U_ξ an immediate basin for f . Then, there exists $k \in \{1, \dots, d - 1\}$ such that U_ξ contains k critical points of f (counting multiplicities), $f|_{U_\xi}$ is a covering map of degree $k + 1$, and U_ξ has exactly k accesses to ∞ . \square*

2. THE CHANNEL DIAGRAM

In the following, by a (*finite*) *graph* we mean a connected topological space Γ homeomorphic to the quotient of a finite disjoint union of closed arcs by an equivalence relation on the set of their endpoints. The arcs are called *edges* of the graph, an equivalence class of endpoints a *vertex*.

Definition 2.1 (Graph Map). *Let $\Gamma_1, \Gamma_2 \subset \mathbb{S}^2$ be two finite graphs and $g : \Gamma_1 \rightarrow \Gamma_2$ continuous. We call g a graph map if it is injective on each edge of Γ_1 and forward and inverse images of vertices are vertices. If the graph map g is a homeomorphism, then we call it a graph homeomorphism.*

In the following, the closure and boundary operators will be understood with respect to the topology of $\widehat{\mathbb{C}}$, unless otherwise stated. Also, we will say that a set $X \subset \widehat{\mathbb{C}}$ is *bounded* if $\infty \notin \overline{X}$.

We say that a Newton map f of degree $d \geq 3$ *satisfies* (\star) if it has the following property:

$$(\star) \quad \begin{cases} \text{if } c \text{ is a critical point of } f \text{ with } c \in B_{\xi_j} \text{ for some} \\ j \in \{1, \dots, d\}, \text{ then } c \text{ has finite orbit.} \end{cases}$$

We omit the easy proof of the following well-known fact.

Lemma 2.2 (Only Critical Point). *Let f be a Newton map that satisfies (\star) and let $\xi \in \mathbb{C}$ be a fixed point of f with immediate basin U_ξ . Then, ξ is the only critical point in U_ξ .* \square

For the following construction let us assume that f satisfies (\star) . It follows that each immediate basin U_ξ has a global Böttcher map $\varphi_\xi : (\mathbb{D}, 0) \rightarrow (U_\xi, \xi)$ with the property that $f(\varphi_\xi(z)) = \varphi_\xi(z^{k_\xi})$ for each $z \in \mathbb{D}$, where $k_\xi - 1 \geq 1$ is the multiplicity of ξ as a critical point of f [Mi2, Theorems 9.1 & 9.3]. The $k_\xi - 1$ radial lines (or *internal rays*) in \mathbb{D} which are fixed under $z \mapsto z^{k_\xi}$ map under φ to $k_\xi - 1$ pairwise disjoint, non-homotopic injective curves $\Gamma_\xi^1, \dots, \Gamma_\xi^{k_\xi-1}$ in U_ξ that connect ξ to ∞ and are each invariant under f . They represent all accesses to ∞ of U_ξ , see Proposition 1.9. Hence if $\xi_1, \dots, \xi_d \in \mathbb{C}$ are the attracting fixed points of f , then the union

$$\Delta := \bigcup_{i=1}^d \bigcup_{j=1}^{k_{\xi_i}-1} \overline{\Gamma_{\xi_i}^j}$$

of these invariant curves over all immediate basins forms a connected and f -invariant graph in $\widehat{\mathbb{C}}$ with vertices at the ξ_i and at ∞ . We call Δ the *channel diagram* of f . The channel diagram records the mutual locations of the immediate basins of f and provides a first-level combinatorial structure to the dynamical plane. Figure 1 shows a Newton map and its channel diagram. The following definition is an axiomatization of the channel diagram.

The following useful observation is a special case of [RS, Theorem 4.8].

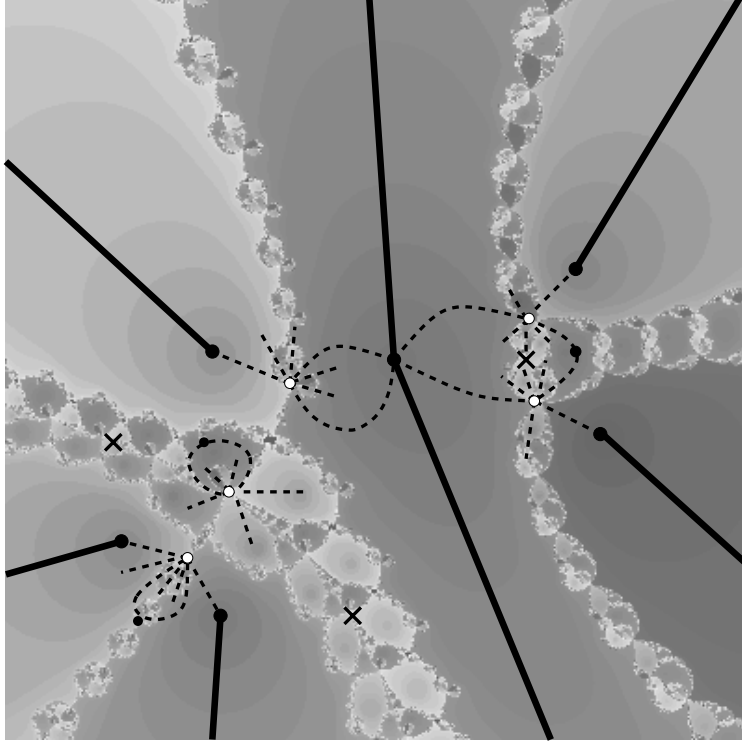


FIGURE 1. A Newton map of degree 6, superimposed with its channel diagram: the solid lines represent the fixed rays of the immediate basins, the black dots correspond to the fixed points (the vertex at ∞ is not visible). The dashed lines show the first preimage of the channel diagram: white circles represent poles, a cross is a free critical point. The Newton map has a pole outside Δ_1 , and the right boundary component of the central immediate basin contains two poles.

Lemma 2.3 (Fixed Points). *Let f be a Newton map and let $D \subset \hat{\mathbb{C}}$ be a closed topological disk such that $\gamma := f(\partial D)$ is a simple closed curve with the property that $\gamma \cap \mathring{D} = \emptyset$. Let V be the unique component of $\hat{\mathbb{C}} \setminus \gamma$ that contains \mathring{D} and let $\{\gamma'_i\}_{i \in I}$ be the collection of boundary components of $f^{-1}(V) \cap D$. Then, the number of fixed points of f in D equals*

$$\sum_{i \in I} \left| \deg(f|_{\gamma'_i} : \gamma'_i \rightarrow \gamma) \right|.$$

In particular, if $f^{-1}(V) \cap D \neq \emptyset$, then D contains a fixed point. \square

REMARK. Since f has no parabolic fixed points, we do not need to take multiplicities of fixed points into account.

Note also that the γ'_i are exactly the components of $f^{-1}(\gamma) \cap D$, except possibly ∂D itself. The boundary is excluded if points in \mathring{D} near ∂D are mapped out of \bar{V} . By the lemma, the only case in which D does not contain a fixed point of f is if all of \mathring{D} is mapped outside of \bar{V} .

The following theorem shows a relation between poles and fixed points outside immediate basins. It considerably sharpens [RS, Corollary 5.2], which states that for an immediate basin U_ξ of a Newton map, every component of $\mathbb{C} \setminus U_\xi$ contains at least one fixed point.

Theorem 2.4 (Fixed Points and Poles). *Let f be a Newton map and U_ξ an immediate basin. If W is a component of $\mathbb{C} \setminus U_\xi$, then the number of fixed points in W equals the number of poles in W , counting multiplicities.*

Proof. Let $d \geq 3$ be the degree of f . If U_ξ does not separate the plane, i.e. if it has only one access to ∞ , then the claim follows trivially: W contains all $d - 1$ finite poles and the $d - 1$ other finite fixed points of f . So suppose in the following that there is a Riemann map $\varphi : (\mathbb{D}, 0) \rightarrow (U_\xi, \xi)$ with $f(\varphi(z)) = \varphi(g(z))$ for some Blaschke product $g : \mathbb{D} \rightarrow \mathbb{D}$ of degree $k := \deg(f|_{U_\xi} : U_\xi \rightarrow U_\xi) \geq 3$.

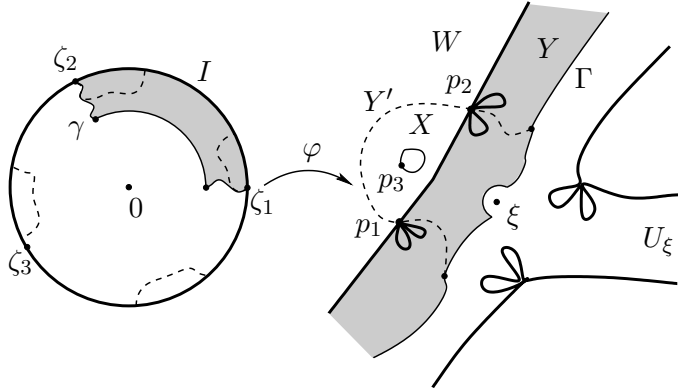


FIGURE 2. In the proof of Theorem 2.4, the construction of $\gamma \subset \mathbb{D}$ is shown on the left for $k = 4$. The dashed curves are the components of $g^{-1}(\gamma)$. The right picture shows the curve $\Gamma \subset U_\xi$. The dashed curve indicates where Γ'_1 differs from Γ .

We may extend g by reflection to a rational function $\hat{g} : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ of degree k whose Julia set equals \mathbb{S}^1 and that has $k - 1$ fixed points $\zeta_1, \dots, \zeta_{k-1} \in \mathbb{S}^1$. These fixed points correspond to the accesses to ∞ of U_ξ . Since \hat{g} fixes \mathbb{D} , the ζ_i have real positive multipliers and since 0 and ∞ attract all of \mathbb{D} and of $\widehat{\mathbb{C}} \setminus \mathbb{D}$, respectively, none of the ζ_i can be attracting or parabolic. Hence they are pairwise distinct and repelling. For each ζ_i , choose a linearizing neighborhood and choose $0 < \varrho < 1$ large enough so that all critical values of \hat{g} in \mathbb{D} have absolute value less than ϱ and so that

the linearizing neighborhoods of all ζ_i intersect the circle at radius ϱ . Let $\gamma \subset \mathbb{D}$ be the unique curve with the following properties, see Figure 2: there are adjacent fixed points $\zeta_j, \zeta_{j+1} \in \mathbb{S}^1$ and injective curves γ_j, γ_{j+1} , so that γ_j connects ζ_j to the circle at radius ϱ and γ_j is a straight line segment in linearizing coordinates of ζ_j ; the same for γ_{j+1} and ζ_{j+1} . Their closures separate the circle at radius ϱ into two arcs. Of those arcs, let γ' be the one for which $\gamma := \gamma_j \cup \overline{\gamma'} \cup \gamma_{j+1}$ has the property that $\varphi(\gamma) \subset U_\xi$ separates W from ξ . Let $\Gamma := \varphi(\gamma) \cup \{\infty\}$. Then, Γ is a simple closed curve in \overline{U}_ξ and contains no critical values, except possibly ∞ .

Let us first suppose that ∞ is not a critical value. Then, every component Γ'_i of $f^{-1}(\Gamma)$ is a simple closed curve and $\deg(f|_{\Gamma'_i} : \Gamma'_i \rightarrow \Gamma)$ equals the number of poles on Γ'_i (here, we do not need to count multiplicities, because we have assumed that f has no critical poles).

Let Γ'_1 be the component of $f^{-1}(\Gamma)$ containing ∞ . We claim that $\Gamma'_1 \cap U_\xi$ consists of two connected components, each of which is an injective curve that connects ∞ to a pole on ∂U_ξ ; call these poles p_1 and p_2 . Indeed, consider the situation in \mathbb{D} -coordinates. Let $I \subset \mathbb{S}^1$ be the arc between ζ_j and ζ_{j+1} that is separated from 0 by γ . Since \mathring{I} contains no fixed points of \hat{g} , $\hat{g}(\overline{I})$ covers $\mathbb{S}^1 \setminus \overline{I}$ exactly once and \overline{I} itself exactly twice. Hence, it is easy to see that $\hat{g}^{-1}(\gamma)$ has exactly two connected components that intersect γ . This proves the claim.

Let Y be the closure of the component of $\widehat{\mathbb{C}} \setminus \Gamma$ that contains W and let Y' be the closure of the component of $\widehat{\mathbb{C}} \setminus \Gamma'_1$ that intersects W in an unbounded set. We distinguish two cases.

If $p_1 = p_2$, then $\Gamma'_1 \subset \overline{U}_\xi$ and $W \subset Y'$. Moreover, Y' is a closed topological disk that contains exactly the same fixed points and poles of f as W . Since $f(\partial Y') \cap \mathring{Y}' = \emptyset$, Lemma 2.3 gives that the number of fixed points in Y' (including ∞) equals the number of poles (again including ∞), because on every component of $f^{-1}(\Gamma)$ in Y' , the degree of f equals the number of poles it contains. Excluding ∞ again, the claim follows.

If $p_1 \neq p_2$, then $\Gamma'_1 \not\subset \overline{U}_\xi$ and $W \not\subset Y'$ (this is the situation pictured in Figure 2). If the set $X := W \setminus Y'$ contains neither poles nor fixed points of f , then we can proceed as before.

Indeed, $\Gamma'_1 \cup U_\xi$ separates X from ∞ and since every fixed point of f is surrounded by its unbounded immediate basin, X cannot contain a fixed point. Now suppose by way of contradiction that X contains a pole p_3 of f . If $p_3 \in \partial X$, then $p_3 \in \partial U_\xi$. But this would imply the existence of an additional pre-fixed point of \hat{g} on I , a contradiction. The other case is that $p_3 \in \mathring{X}$. Then, there exists a component Γ'_2 of $f^{-1}(\Gamma)$ in \mathring{X} . Let D be the bounded disk bounded by Γ'_2 . We may assume without loss of generality that there is no component of $f^{-1}(\Gamma)$ separating D from ∂X . Since points in Y' near Γ'_1 are mapped into Y under f , points in X near Γ'_1 are mapped out of Y , and it follows that again $f(D) \cap Y \neq \emptyset$. Now, \overline{D} contains a fixed point by Lemma 2.3. This is a contradiction.

In the remaining case that ∞ is a critical value, we perturb f slightly to avoid that situation. Since poles and fixed points of f move continuously under perturbation, and Γ does too, this does not change the count. Note that while U_ξ and W might move discontinuously, this does not pose a problem because we have actually counted poles and fixed points in Y' , after having established that we do not lose anything by this replacement. \square

Corollary 2.5 (Fixed Points in Complement). *Let f be a Newton map that satisfies (\star) and Δ be the channel diagram of f . For a component V of $\widehat{\mathbb{C}} \setminus \Delta$ let p be the number of poles of f in V , counting multiplicities. Then $\partial V \cap \mathbb{C}$ contains $p + 1$ fixed points.*

Proof. If V is the only component of $\widehat{\mathbb{C}} \setminus \Delta$, the claim follows trivially. If ξ is the only fixed point on ∂V whose immediate basin U_ξ separates the plane, then the claim follows directly from Theorem 2.4. Indeed, let in this case R_1, R_2 be the fixed internal rays of U_ξ that are on ∂V and let V_1 be the component of $\mathbb{C} \setminus (R_1 \cup R_2 \cup \{\xi\})$ such that $V \subset V_1$. Then, V_1 also contains p poles and by Theorem 2.4, V_1 contains p fixed points. Since $\xi \in \partial V$ as well, the claim follows.

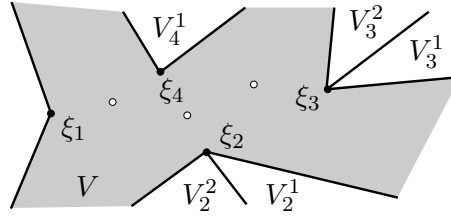


FIGURE 3. In Corollary 2.5, the open set V is bounded by parts of Δ . The black dots are fixed points, the white dots represent poles. The V_j^i may well contain further structure of Δ .

Now suppose that $\xi_1, \dots, \xi_k \in \partial V$ are the fixed points on ∂V whose immediate basins separate the plane. Let R_1, R_2 be the fixed internal rays of U_{ξ_1} on ∂V and let V_1 be as above. Let m be the number of poles in V_1 . As before, it follows that V_1 contains m fixed points. Let $m' = m - p$. For $j = 2, \dots, k$, denote by $V_j^1, \dots, V_j^{i_j}$ all complementary components of the closures of the fixed internal rays of U_{ξ_j} that do not contain V . By Theorem 2.4, each V_j^i contains as many poles as fixed points, hence all V_j^i combined contain m' poles and m' fixed points. Hence, $V_1 \setminus (\bigcup_{j=2}^k \bigcup_{\ell=1}^{i_j} V_j^\ell)$ contains $m - m' = p$ fixed points. The claim now follows, because including ξ_1 , $\partial V \cap \mathbb{C}$ contains $p + 1$ fixed points. \square

Corollary 2.6 (Existence of Shared Poles). *Let f be a Newton map and Δ be the channel diagram of f . If V is a component of $\widehat{\mathbb{C}} \setminus \Delta$, then there*

is at least one pair of fixed points $\xi_1, \xi_2 \in \partial V \cap \mathbb{C}$ such that ∂U_{ξ_1} and ∂U_{ξ_2} intersect in a pole.

Proof. Let U_ξ be an immediate basin. Clearly, the components of $\partial U_\xi \cap \mathbb{C}$ are separated by the accesses to ∞ . If $\partial U_\xi \cap \mathbb{C}$ has only one component, then in the conjugate dynamics $\hat{g}|_{\mathbb{D}}$ we have $\hat{g}(\mathbb{S}^1) = \mathbb{S}^1$ therefore S^1 contains pre-fixed points of \hat{g} . In case $\partial U_\xi \cap \mathbb{C}$ consists of more than one component we have seen that in the conjugate dynamics $\hat{g}|_{\mathbb{D}}$, for every arc $I \subset \mathbb{S}^1$ between two fixed points, $\hat{g}(I) = \mathbb{S}^1$. Therefore, I contains pre-fixed points of \hat{g} . Since poles and ∞ are accessible boundary points of U_ξ , we conclude that every component of $\partial U_\xi \cap \mathbb{C}$ contains at least one pole. By Corollary 2.5, there has to be at least one pole in V that is on the boundary of at least two immediate basins. \square

REMARK. Figure 1 shows that a component of $\partial U_\xi \cap \mathbb{C}$ may contain more than one pole.

Note also that a simple pole is on the boundary of at most two immediate basins, because otherwise f cannot preserve the cyclic order of the immediate basins near that pole. This was first observed by Janet Head [He].

3. PROOF OF THEOREM 1.4

Let $\xi_1, \dots, \xi_d \in \mathbb{C}$ be the finite fixed points of f and let U'_0 be a component of some B_{ξ_i} . Let Δ be the channel diagram of f . Recall that it consists of invariant rays within the U_{ξ_i} that connect the ξ_i to ∞ . Denote by Δ_n the connected component of $f^{-n}(\Delta)$ that contains Δ (with this convention, $\Delta = \Delta_0$). Every edge of Δ_n is then an internal ray of a component of some B_{ξ_i} , while every vertex is a preimage of a ξ_i , or a pole or prepole or ∞ .

To prove Theorem 1.4, it suffices to show that there exists $n \in \mathbb{N}$ such that Δ_n contains all poles of f : then, every pole of f can be connected to ∞ through a finite chain of internal rays in the B_{ξ_i} , and hence through a finite chain of components of the basins. By induction, each prepole is in Δ_m for sufficiently large m . Since ∞ is on the boundary of every immediate basin, $\partial U'_0$ contains a prepole. This finishes the argument.

It remains to show that there exists $n \in \mathbb{N}$ such that Δ_n contains all poles of f . If Δ_1 contains all poles of f , then we are done. So assume in the following that there exists a component C_1 of $f^{-1}(\Delta)$ such that $C_1 \cap \Delta_1 = \emptyset$ (Figure 1 shows that this does occur). Equivalently, we may assume that there exists a component V_0 of $\widehat{\mathbb{C}} \setminus \Delta$ and a component V_1 of $f^{-1}(V_0)$ such that V_1 is multiply connected. Then, we choose C_1 so that it intersects ∂V_1 .

Denote by C_n the component of $f^{-n}(\Delta)$ containing C_1 . We will assume that $C_n \cap \Delta_n = \emptyset$ for all $n \in \mathbb{N}$ (otherwise we would be done). We will lead this assumption to a contradiction.

Lemma 3.1 (Preimage Inside). *With the above notation, $V_1 \subset V_0$.*

Proof. Since $\Delta \subset f^{-1}(\Delta)$, we either have $V_1 \subset V_0$ or $V_1 \cap V_0 = \emptyset$. In the latter case, let $\gamma \subset V_0$ be a simple closed curve near ∂V_0 that surrounds all

critical values within V_0 (note that V_0 contains critical values, because it is the image of the multiply connected domain V_1). Then, $f^{-1}(\gamma) \cap V_1$ consists of several nested and non-contractible (in V_1) simple closed curves (at least two, since otherwise V_0 would be a homeomorphic image of V_1 and hence multiply connected). Let γ' be the outermost of them, D be the bounded component of $\widehat{\mathbb{C}} \setminus \gamma'$ and V the complement component of $\widehat{\mathbb{C}} \setminus \gamma$ containing D . The choice of γ' being the outermost preimage of γ in V_1 implies that $f^{-1}(V) \cap D \neq \emptyset$. Hence by Lemma 2.3 we obtain that \overline{D} contains a fixed point of f . This is a contradiction, because \overline{D} is separated from Δ and all fixed points are contained in Δ . \square

Lemma 3.2 (Preimage Unbounded). *Let $n \in \mathbb{N}$ and suppose that W is an unbounded component of $\widehat{\mathbb{C}} \setminus \Delta_n$. If W' is a component of $f^{-1}(W)$ with $W' \subset W$, then W' is unbounded.*

Proof. Let ξ_1, \dots, ξ_k be the attracting fixed points of f in ∂W . By Lemma 2.2, f has Böttcher coordinates near each ξ_i . For $i \in \{1, \dots, k\}$, choose a neighborhood B_i of ξ_i that has the following properties: B_i contains no critical values except ξ_i ; in Böttcher coordinates, B_i is a round disk centered at 0, small enough so that $\Delta_{n+1} \cap B_i$ consists of radial lines; f is conjugate to $z \mapsto z^{k_i}$ on B_i . Since Δ_n is a graph, there exists a simple closed curve $\gamma \subset \partial W \subset \Delta_n$ that surrounds W and thus W' . By possibly modifying γ within the B_i , we may assume that in Böttcher coordinates, every point $z \in \gamma \cap B_i$ is either in Δ_n or on a circle of constant radius centered at 0. Let D be the component of $\widehat{\mathbb{C}} \setminus \gamma$ that intersects W . Then, $\overline{D} \cap \gamma \subset \bigcup_{i=1}^k B_i$.

Now suppose by way of contradiction that $W' \subset W$ is bounded and let γ' be the outermost (in \mathbb{C}) simple closed curve in $f^{-1}(\gamma)$ that intersects $\partial W'$. Let D' be the component of $\mathbb{C} \setminus \gamma'$ that is contained in D . Observe that $D' \cap \gamma = \emptyset$: γ intersects D' at most in the B_i . But there, γ was chosen in such a way that γ' is strictly further away from ξ than γ , so in particular, γ and D' are disjoint. Now, either $f(D') \subset D$ or $\overline{D'}$ contains another component of $f^{-1}(\gamma)$. In both cases, Lemma 2.3 shows that $\overline{D'}$ contains a fixed point of f . This is a contradiction, because γ was constructed in such a way that $\overline{D'}$ does not contain an attracting fixed point, while ∞ is not in $\overline{D'}$ by assumption. \square

Corollary 3.3 (Free Pole in Unbounded Nest). *Suppose that $\Delta_n \cap C_n = \emptyset$ for all $n \in \mathbb{N}$. Then for each $n \geq 1$, there exists an unbounded and multiply connected component V_n of $f^{-n}(V_0)$ such that $V_n \subset V_{n-1}$ and $C_n \cap \partial V_n \neq \emptyset$, while the component of ∂V_n that contains ∞ is in Δ_n .*

Proof. By definition, V_1 is multiply connected and $\partial V_1 \cap C_1 \neq \emptyset$. By Lemma 3.1, $V_1 \subset V_0$ and by Lemma 3.2, V_1 is unbounded.

Now suppose by induction that V_n has the claimed properties. Clearly, every component of $f^{-1}(V_n)$ is multiply connected. Since $\infty \in \partial V_n$, there exists a component V_{n+1} of $f^{-1}(V_n)$ such that $\partial V_{n+1} \cap C_{n+1} \neq \emptyset$, since $C_1 \subset$

C_{n+1} and C_1 contains a pole by construction. Using the same argument as in Lemma 3.1 we obtain that $V_{n+1} \subset V_n$.

Let W be the component of $\widehat{\mathbb{C}} \setminus \Delta_n$ containing V_n and let W' be the component of $f^{-1}(W)$ containing V_{n+1} . If V_{n+1} was bounded, then so would be W' , contradicting Lemma 3.2. Since Δ_{n+1} is the only unbounded component of $f^{-(n+1)}(\Delta)$, the component of ∂V_{n+1} that contains ∞ is in Δ_{n+1} and we are done. \square

We call the unbounded component of ∂V_n the *outer boundary* and denote it with B_n .

Recall that to finish the proof of Theorem 1.4 in case of the finiteness condition (\star) , it suffices to show the following.

Theorem 3.4 (Poles Connect to ∞). *There exists $n \in \mathbb{N}$ such that Δ_n contains all poles of f .*

Proof. Suppose by way of contradiction that $C_n \cap \Delta_n = \emptyset$ for all $n \in \mathbb{N}$. Suppose first that for large enough n , V_n surrounds only one pole, and this pole is in C_n .

For any $n \in \mathbb{N}$, we call a closed arc in B_n a *bridge* if it connects two distinct finite fixed points ξ_1 and ξ_2 and contains no other fixed points, in particular not ∞ . The *length* of a bridge is the number of edges of Δ_n it consists of. We say that two distinct finite fixed points $\xi_1, \xi_2 \in B_n$ are *adjacent* if there exists a bridge in B_n connecting them.

By Corollary 3.3, all V_n are unbounded and hence have at least one access to ∞ . The number of fixed points in ∂V_n is non-increasing: if there were a fixed point in ∂V_n , then it would also have been in its image ∂V_{n-1} . Moreover, every unbounded edge of B_n is contained in Δ and connects ∞ to a finite fixed point. It now follows from Definition 4.4 (4), that each B_n contains at least two finite fixed points. Hence, the number of fixed points and accesses in ∂V_n cannot shrink infinitely often and there exists a minimal n_0 such that for $n \geq n_0$, B_n and B_{n_0} contain the same fixed points and V_n and V_{n_0} have the same accesses to ∞ . Therefore, two distinct finite fixed points $\xi_1, \xi_2 \in B_n$ are adjacent for $n \geq n_0$ if and only if they are adjacent in B_{n_0} : a bridge in B_n between them that does not exist in B_{n_0} would separate another fixed point on B_{n_0} (or an access to ∞ of V_{n_0}) from V_n ; the other direction is trivial.

Claim 1: If ξ is a finite fixed point that is in B_n for all $n \in \mathbb{N}$, then ξ is adjacent to at most one finite fixed point.

Proof of Claim 1: Let $\varphi : \mathbb{D} \rightarrow U_\xi$ be a Böttcher map that conjugates f to $g(z) = z^k$ for some $k \geq 2$ and let $R_1, \dots, R_{k-1} \subset \mathbb{D}$ be the fixed internal rays of g . For all $n \in \mathbb{N}$, $\varphi^{-1}(V_n \cap U_\xi)$ equals a sector S_n of $\mathbb{D} \setminus \bigcup_{i=1}^{k-1} \overline{g^{-n}(R_i)}$, such that $S_{n+1} \subset S_n$ and $g(S_{n+1}) = S_n$. It is easy to see that this can only happen if one of the boundary arcs of all S_n is a fixed ray. Indeed, the intersection $\bigcap_{n=1}^{\infty} \overline{S_n}$ must be a single ray R . Its image $g(R) \subset \bigcap_{n=1}^{\infty} \overline{S_n}$,

hence $g(R) = R$. Since no fixed ray can be inside an S_n , it must be part of the boundary of all S_n .

It follows that ξ and ∞ are connected in all B_n by a fixed edge, and at most one other edge that ends at ξ can be part of a bridge in B_n . \square

Observe also that B_{n_0} contains a bridge: by minimality of n_0 , there is a finite fixed point on B_{n_0-1} or an access to ∞ of V_{n_0-1} that is separated from V_{n_0} by an arc in B_{n_0} . Possibly by extending this arc along B_n , we find a bridge $X_{n_0} \subset B_{n_0}$ between two finite fixed points (this is possible because the extension must hit a finite fixed point before ∞). If $n_0 = 1$, then B_1 contains a bridge X_{n_0} between two finite fixed points by Corollary 2.6. Let ξ_1 and ξ_2 be the endpoints of X_{n_0} .

By induction on $n \geq n_0$, let X_{n+1} be the shortest arc in $f^{-1}(X_n)$ that ends at ξ_1 and connects ξ_1 to a point $\xi'_2 \in f^{-1}(\{\xi_2\})$ within B_{n+1} . If $\xi'_2 = \xi_2$, then X_{n+1} is a bridge as well, otherwise we say that the bridge X_{n_0} *disconnects* at time $n + 1$.

Note that if X_{n+1} is a bridge, then the length of X_{n+1} can only be greater than the one of X_n if X_{n+1} contains a pole. Hence the length is constant for all sufficiently large n .

Claim 2: The bridge X_{n_0} disconnects after finitely many pull-backs.

Proof of Claim 2: Suppose that X_n is a bridge from ξ_1 to ξ_2 for all $n \geq n_0$. Near the endpoints ξ_1 and ξ_2 , X_n consists of internal rays $R_1^n \subset U_{\xi_1}$ and $R_2^n \subset U_{\xi_2}$, respectively. Hence, we can express X_n as $R_1^n \cup R_2^n \cup X'_n$, where X'_n consists of a bounded number of edges of Δ_n by the previous considerations. Considering the situation in Böttcher coordinates as in the previous claim, we see that R_1^n converges to a fixed ray $R_1^0 \subset U_{\xi_1}$, and R_2^n converges to a fixed ray $R_2^0 \subset U_{\xi_2}$. Let r_n be the non-fixed endpoint of R_1^n . Then, $r_n \rightarrow \infty$ as $n \rightarrow \infty$.

Since ∞ is a repelling fixed point with multiplier $\lambda > 1$, there exists a branched covering map $\psi : (\mathbb{C}, 0) \rightarrow (\widehat{\mathbb{C}} \setminus \{0\}, \infty)$ such that $f(\psi(z)) = \psi(\lambda z)$ for all $z \in \mathbb{C}$ (we may assume without loss of generality that $0 \notin \overline{V_0}$). Moreover, there exist a neighborhood W of ∞ and a holomorphic branch φ of ψ^{-1} on W [Mi2, Corollary 8.10]. Let N be sufficiently large so that for all $n \geq N$, $r_n \in W$ and $\overline{X'_n}$ contains no critical values of f . Then, $\overline{X'_{n+1}}$ is the lift of $\overline{X'_n}$ under the branch of f^{-1} that maps r_n to r_{n+1} . Observe that a point $z \in \mathbb{C}$ is a critical point of ψ if and only if f has a critical point at $f^{\circ j}(\psi(z/\lambda^m))$ for some $j \leq m$, where m is chosen large enough so that $z/\lambda^m \in \varphi(W)$. Therefore, none of the X'_n for $n \geq N$ contain a critical value of ψ and we can pull back X'_n under the branch of ψ^{-1} that maps r_n to $\varphi(r_n)$. Since f commutes with multiplication by λ , this pull-back operation commutes with division by λ and it follows that X'_n converges uniformly to ∞ as a set as $n \rightarrow \infty$. Therefore, $X_n \rightarrow \overline{R_1^0} \cup \overline{R_2^0}$ uniformly as a set as $n \rightarrow \infty$. (Note that since X_n is connected by assumption, we could have as well argued starting with the endpoint of R_2^n .)

Let U_n be an unbounded component of $\bar{C} \setminus \overline{R_0^1 \cup R_0^2 \cup X_n}$, that contains C_n . Then $\bar{C} \setminus C_n$ has bounded components of some positive area, on other hand $\text{area}(U_n) \rightarrow 0$, because its boundary converges to an arc $R_0^1 \cup R_0^2$. This is a contradiction. \square

We can now finish the proof of Theorem 3.4.

If X_{n_0} disconnects at time $n+1$, then consider its endpoint $\xi'_2 \in f^{-1}(\{\xi_2\})$. If $\xi'_2 \notin \Delta_1$, then it must by assumption be in C_1 . Thus, X_{n+1} connects C_n to Δ . This means that $C_{n+1} \subset \Delta_{n+1}$ and we are done. If $\xi'_2 \in \Delta_1$, we can extend X_{n+1} to a bridge by connecting ξ'_2 to a pole $p \in V_n$ and to another fixed point ξ_3 , different from ξ_1 (there always exists a bridge in ∂V_{n+1} connecting two different fixed points) within B_{n+1} . Claim 1 implies that $\xi_3 \in \{\xi_1, \xi_2\}$, without loss of generality we assume that $\xi_3 = \xi_2$.

But then we have found a new bridge between ξ_1 and ξ_2 . According to Claim 2 the new bridge also disconnects after taking some pullbacks. It follows from Claim 1 that $V_{n+1} \cap U_{\xi_2} \subset V_n \cap U_{\xi_2}$, therefore each time the new constructed bridge connecting ξ_1 to ξ_2 in ∂V_n disconnects we must encounter a new pole on the boundary of V_n . However, there are only finitely many poles of f and we get a contradiction. Therefore at some level n , $\xi'_2 \in C_1$ and X_n connects C_n to Δ .

If all V_n surround several bounded components of $f^{-1}(\Delta)$, our arguments show that at least one of them is connected to Δ_k for some $k \in \mathbb{N}$. To finish the proof, it suffices to show that at time k , a new bridge is created that connects ξ_1 to some finite fixed point ξ_3 . Then we can continue by induction. To see that this bridge exists, let C_1 be the component of $f^{-1}(\Delta)$ that was connected to Δ_k by X_k . Observe that C_1 also contains a preimage of ξ_1 . We can extend the arc X_k from ξ_1 to $\xi'_2 \in C_1$ to this preimage of ξ_1 and then further to another preimage of ξ_2 etc., until we arrive at a finite fixed point, say ξ_3 . As before, ξ_3 might equal ξ_2 if X_k contains a multiple pole. The above arguments apply to all components of $\hat{\mathbb{C}} \setminus \Delta$ that surround a pole. \square

This finishes the proof of Theorem 1.4 under the finiteness condition (\star) . In general, we can use a straightforward surgery construction to bring any Newton map into the desired form: all we require is finiteness of critical orbits in all basins of roots; there may well be infinite (or even periodic or preperiodic) critical orbits in the Julia set, or in attracting, parabolic, or Siegel components of the Fatou set.

Within any immediate basin U_ξ , we may replace the attracting dynamics (which may involve several critical points converging to the root ξ) by dynamics modeled after $z \mapsto z^k$ within the unit disk. This surgery procedure does not affect the correctness of Theorem 1.4. A similar procedure can assure that all critical points in the entire basin B_ξ of ξ land on the fixed point ξ after finitely many steps. Details are standard and thus omitted; compare for example Shishikura [S]. This proves Theorem 1.4 in the general case.

4. THE NEWTON GRAPH OF A NEWTON MAP

In this section, we define abstract Newton graphs and use Theorem 1.4 to show that every postcritically fixed Newton map f generates a unique abstract Newton graph in a natural way. We make some references to Thurston theory, which is discussed in more detail in Section 5.

4.1. Extending Maps on Finite Graphs. The channel diagram motivates the definition of a Newton graph. For this, we first need to introduce some notation regarding maps on imbedded graphs and their extensions to \mathbb{S}^2 , compare [BFH, Chapter 6]. We assume in the following without explicit mention that all graphs are imbedded into \mathbb{S}^2 .

Definition 4.1 (Regular Extension). *Let $g : \Gamma_1 \rightarrow \Gamma_2$ be a graph map. An orientation-preserving branched covering map $\bar{g} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is called a regular extension of g if $\bar{g}|_{\Gamma_1} = g$ and \bar{g} is injective on each component of $\mathbb{S}^2 \setminus \Gamma_1$.*

Lemma 4.2 (Isotopic Graph Maps). [BFH, Corollary 6.3] *Let $g, h : \Gamma_1 \rightarrow \Gamma_2$ be two graph maps that coincide on the vertices of Γ_1 such that if $\gamma \subset \Gamma_1$ is an edge, then $g(\gamma) = h(\gamma)$ as a set. Suppose that g and h have regular extensions $\bar{g}, \bar{h} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$. Then there exists a homeomorphism $\psi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$, isotopic to the identity relative the vertices of Γ_1 , such that $\bar{g} = \bar{h} \circ \psi$. \square*

Let $g : \Gamma_1 \rightarrow \Gamma_2$ be a graph map. For the next proposition, we will assume without loss of generality that each vertex v of Γ_1 has a neighborhood $U_v \subset \mathbb{S}^2$ such that all edges of Γ_1 that enter U_v terminate at v ; we may also assume that U_v is a round disk of radius r centered at v , where r is chosen sufficiently small so that U_{v_1} and U_{v_2} are disjoint for different vertices $v_1 \neq v_2$, all edges entering U_v are straight in U_v and that $g|_{U_v}$ is length-preserving. We make analogous assumptions for Γ_2 . Then, we can extend g to each U_v as in [BFH]: for a vertex $v \in \Gamma_1$, let γ_1 and γ_2 be two adjacent edges ending there. In polar coordinates centered at v , these are straight lines with arguments, say, θ_1, θ_2 such that $0 < \theta_2 - \theta_1 \leq 2\pi$ (if v is an endpoint of Γ_1 , then set $\theta_1 = 0, \theta_2 = 2\pi$). In the same way, choose arguments θ'_1, θ'_2 for the image edges in $U_{g(v)}$ and extend g to a map \tilde{g} on $\Gamma_1 \cup \bigcup_v U_v$ by setting

$$(\varrho, \theta) \mapsto \left(\varrho, \frac{\theta'_2 - \theta'_1}{\theta_2 - \theta_1} \cdot \theta \right),$$

where (ϱ, θ) are polar coordinates in the sector bounded by the rays at θ_1 and θ_2 . In other words, sectors are mapped onto sectors in an orientation-preserving way. Then, the following holds.

Proposition 4.3 (Regular Extension). [BFH, Proposition 6.4] *The map $g : \Gamma_1 \rightarrow \Gamma_2$ has a regular extension if and only if for every vertex $y \in \Gamma_2$ and every component U of $\mathbb{S}^2 \setminus \Gamma_1$, the extension \tilde{g} is injective on*

$$\bigcup_{v \in g^{-1}(\{y\})} U_v \cap U.$$

In this case, the regular extension \bar{g} may have critical points only at the vertices of Γ_1 . \square

4.2. The Newton Graph. Let us first define an abstract channel diagram which plays the basement role for the main concept of an abstract Newton graph.

Definition 4.4 (Abstract Channel Diagram). *An abstract channel diagram of degree $d \geq 3$ is a graph $\Delta \subset \mathbb{S}^2$ with vertices v_0, \dots, v_d and edges e_1, \dots, e_l that satisfies the following properties:*

- (1) $l \leq 2d - 2$;
- (2) each edge joins v_0 to a v_i , $i > 0$;
- (3) each v_i is connected to v_0 by at least one edge;
- (4) if e_i and e_j both join v_0 to v_k , then each connected component of $\mathbb{S}^2 \setminus \overline{e_i \cup e_j}$ contains at least one vertex of Δ .

We say that an abstract channel diagram Δ is *realized* if there exist a Newton map with channel diagram $\hat{\Delta}$ and a graph homeomorphism $h : \Delta \rightarrow \hat{\Delta}$ that preserves the cyclic order of edges at each vertex.

Lemma 4.5. *The channel diagram $\hat{\Delta}$ of a Newton map f , constructed in section 2 is an abstract channel diagram.*

Proof. By construction, $\hat{\Delta}$ has at most $2d - 2$ edges and it satisfies the properties (2) and (3) of the definition 4.4. Finally, $\hat{\Delta}$ satisfies (4), because for any immediate basin U_ξ of f , every component of $\mathbb{C} \setminus U_\xi$ contains at least one fixed point of f [RS, Corollary 5.2] (see also Theorem 2.4). \square

Now, with these preparations, we are ready to introduce the concept of an abstract Newton graph. It turns out that it carries enough information to uniquely characterize postcritically fixed Newton maps.

Definition 4.6 (Abstract Newton Graph). *Let $\Gamma \subset \mathbb{S}^2$ be a connected graph, Γ' the set of its vertices and $g : \Gamma \rightarrow \Gamma$ a graph map. The pair (Γ, g) is called an abstract Newton graph if it satisfies the following conditions:*

- (1) There exists $d_\Gamma \geq 3$ and an abstract channel diagram $\Delta \subsetneq \Gamma$ of degree d_Γ such that g fixes each vertex and each edge of Δ .
- (2) If v_0, \dots, v_{d_Γ} are the vertices of Δ , then $v_i \in \overline{\Gamma \setminus \Delta}$ if and only if $i \neq 0$. Moreover, there are exactly $\deg_{v_i}(g) - 1 \geq 1$ edges in Δ that connect v_i to v_0 for $i \neq 0$, where $\deg_x(g)$ denotes the local degree of g at $x \in \Gamma'$.
- (3) $\sum_{x \in \Gamma'} (\deg_x(g) - 1) = 2d_\Gamma - 2$.
- (4) There exists $N_\Gamma \in \mathbb{N}$ such that $g^{\circ N_\Gamma}(\Gamma) \subset \Delta$, where N_Γ is minimal such that $g^{\circ(N_\Gamma-1)}(x) \in \Delta$ for all $x \in \Gamma'$ with $\deg_x(g) > 1$.
- (5) The graph $\overline{\Gamma \setminus \Delta}$ is connected.

- (6) For every vertex $y \in \Gamma'$ and every component U of $\mathbb{S}^2 \setminus \Gamma$, the extension \tilde{g} is injective on

$$\bigcup_{v \in g^{-1}(\{y\})} U_v \cap U.$$

- (7) Γ equals the component of $\bar{g}^{-N_\Gamma}(\Delta)$ that contains Δ .

If (Γ, g) is an abstract Newton graph, g can be extended to a branched covering map $\bar{g} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ by (6) and Proposition 4.3. We use this implicitly in (7). Condition (3) and the Riemann-Hurwitz formula ensure that \bar{g} has degree d_Γ . An immediate consequence of Lemma 4.2 is that \bar{g} is unique up to Thurston equivalence.

We say that two abstract Newton graphs (Γ_1, g_1) and (Γ_2, g_2) are *equivalent* if there exists a graph homeomorphism $h : \Gamma_1 \rightarrow \Gamma_2$ that preserves the cyclic order of edges at each vertex of Γ_1 and conjugates g_1 to g_2 .

Now we are ready to prove our first main result. Recall that for a Newton map f with channel diagram Δ , Δ_n denotes the component of $f^{-n}(\Delta)$ that contains Δ .

Proof of Theorem 1.5. Let Δ be the channel diagram of f . First observe that Δ connects every fixed point of f to ∞ . Since f is postcritically fixed, each critical point of f is connected to some prepole by an iterated preimage of Δ . Theorem 3.4 shows that there exists $n \in \mathbb{N}$ such that Δ_n contains all poles of f . Since f is postcritically fixed, it follows by induction that there exists a minimal $N \in \mathbb{N}$ such that Δ_{N-1} contains all critical points of f .

It is easy to see that (Δ_N, f) satisfies all conditions of Definition 4.6 except possibly (5). Note that if all critical points are in Δ_{N-1} , we need to pull back one more step to ensure that condition (3) is satisfied.

To show (5), suppose by way of contradiction that the bounded set $\overline{\Delta_{N-1}} \setminus \Delta$ is not connected. Then, there exists an unbounded component V of $\widehat{\mathbb{C}} \setminus \Delta_{N-1}$ that separates the plane, i.e. V has at least two accesses to ∞ . Let W be a neighborhood of ∞ that is a round disk in linearizing coordinates and satisfies $W \cap \Delta_{N-1} \subset \Delta$. Let V_1, \dots, V_k be the components of $V \cap W$. Then, f acts injectively on each V_i and there exists a branch g_i of f^{-1} that maps V_i into itself (recall that ∞ is a repelling fixed point of f , so it is attracting for the g_i). By assumption, V is simply connected and contains no critical values of f , so the g_i extend to all of V by holomorphic continuation on lines. Since $\Delta_{N-1} \subset \Delta_N$, we get $g_i(V) \subset V$ for all i . If there are $i \neq j$ such that $g_i(V) \cap g_j(V) \neq \emptyset$, then it follows that $g_i = g_j$ and we have found a holomorphic self-map of V for which $\infty \in \partial V$ is attracting through two distinct accesses, contradicting the Denjoy-Wolff theorem [Mi2, Theorem 5.4].

Hence, the $g_i(V)$ are pairwise disjoint and if $w \in \partial g_i(V)$ for some i , then $f(w) \in \partial V$, for otherwise the map g_i would be defined in a neighborhood of $f(w)$. Hence $w \in f^{-1}(\Delta_{N-1})$ and since all $g_i(V)$ are open disks, we even

get that $w \in \Delta_N$. It follows that no component of $\widehat{\mathbb{C}} \setminus \Delta_N$ has more than one access to ∞ , and $\overline{\Delta_N \setminus \Delta}$ is connected.

To prove the last claim, suppose that there exists a graph homeomorphism $h : \Delta_{1,N} \rightarrow \Delta_{2,N}$ such that $h(f_1(z)) = f_2(h(z))$ for all $z \in \Delta_{1,N}$. Since all complementary components of $\Delta_{1,N}$ are disks, we can extend h to a homeomorphism $\bar{h} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ that conjugates f_1 to f_2 up to isotopy relative $\Delta_{1,N}$ [BFH, Chapter 6]. By Theorem 5.6, f_1 and f_2 are conjugate by a Möbius transformation that fixes ∞ . \square

5. ABSTRACT NEWTON GRAPHS ARE REALIZED

In this section, we recall some fundamental notions of Thurston's characterization of rational maps. Then, we use Thurston's theorem to show that every abstract Newton graph is realized by a postcritically fixed Newton map, which is unique up to affine conjugation.

5.1. Thurston's Criterion For Marked Branched Coverings. Thurston's theorem provides a necessary and sufficient condition on the existence of a rational map with certain combinatorial behavior in terms of linear maps generated from a (potentially very large) collection of simple closed curves.

The notations and results in this section are based on [DH] and [PT]. Before we can state Thurston's criterion, we need several definitions. Recall that for a branched covering map $g : \mathbb{S}^2 \rightarrow \mathbb{S}^2$, P_g denotes the postcritical set.

Definition 5.1 (Marked Branched Covering). *A marked branched covering is a pair (g, X) , where $g : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is a postcritically finite branched covering map and X is a finite set containing P_g such that $g(X) \subset X$.*

Definition 5.2 (Thurston Equivalence). *Let (g, X) and (h, Y) be two marked branched coverings. We say that they are (Thurston) equivalent if there are two homeomorphisms $\varphi_0, \varphi_1 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ such that*

$$\varphi_0 \circ g = h \circ \varphi_1$$

and there exists an isotopy $\Phi : [0, 1] \times \mathbb{S}^2 \rightarrow \mathbb{S}^2$ with $\Phi(0, \cdot) = \varphi_0$ and $\Phi(1, \cdot) = \varphi_1$ such that $\Phi(t, \cdot)|_X$ is constant in $t \in [0, 1]$ with $\Phi(t, X) = Y$.

If (g, X) is a marked branched covering and γ a simple closed curve in $\mathbb{S}^2 \setminus X$, then the set $g^{-1}(\gamma)$ is a disjoint union of simple closed curves.

Definition 5.3 (Multicurve). *Let (g, X) be a marked branched covering. We say that a simple closed curve $\gamma \subset \mathbb{S}^2$ is a simple closed curve in (\mathbb{S}^2, X) if $\gamma \subset \mathbb{S}^2 \setminus X$. It is called peripheral if there exists a component of $\mathbb{S}^2 \setminus \gamma$ that intersects X in at most one point, and non-peripheral otherwise.*

Two simple closed curves γ_1, γ_2 in (\mathbb{S}^2, X) are called isotopic (relative X) (write $\gamma_1 \simeq \gamma_2$) if there exists a continuous one-parameter family γ_t , $t \in [1, 2]$, of such curves joining γ_1 to γ_2 . We denote the isotopy class of γ_1 by $[\gamma_1]$.

A finite set $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ of disjoint, non-peripheral and pairwise non-isotopic simple closed curves in (\mathbb{S}^2, X) is called a multicurve.

Definition 5.4 (Irreducible Thurston Obstruction). *Let (g, X) be a marked branched covering and Γ a multicurve. Denote by \mathbb{R}^Γ the real vector space spanned by the isotopy classes of the curves in Γ . Then, we associate to Γ its Thurston transformation $g_\Gamma : \mathbb{R}^\Gamma \rightarrow \mathbb{R}^\Gamma$ by specifying its action on representatives $\gamma \in \Gamma$ of basis elements:*

$$(1) \quad g_\Gamma(\gamma) := \sum_{\gamma' \subset g^{-1}(\gamma)} \frac{1}{\deg(g|_{\gamma'} : \gamma' \rightarrow \gamma)} [\gamma'] .$$

The sum is taken to be zero if there are no preimage components isotopic to a curve in Γ .

The linear map given by equation (1) is represented by a square matrix with non-negative entries and thus its largest eigenvalue $\lambda(\Gamma)$ is real and non-negative by the Perron-Frobenius theorem.

A square matrix $A_{i,j} \in \mathbb{R}^{n \times n}$ is called irreducible if for each (i, j) , there exists $k \geq 0$ such that $(A^k)_{i,j} > 0$. We say that the multicurve Γ is irreducible if the matrix representing g_Γ is.

An irreducible multicurve Γ is called an irreducible (Thurston) obstruction if $\lambda(\Gamma) \geq 1$.

Definition 5.5 (Hyperbolic Orbifold). *Let (g, X) be a marked branched covering of degree $d > 1$. The orbifold O_g is a pair (g, v_g) , where $v_g : X \rightarrow \mathbb{N} \cup \{\infty\}$ is the smallest function such that $v_g(x)$ is a multiple of $v_g(y) \deg_y g$ for each $y \in g^{-1}(\{x\})$.*

The orbifold O_g is called hyperbolic if the Euler characteristic

$$\chi(O_g) = 2 - \sum_{x \in X} \left(1 - \frac{1}{v_g(x)} \right)$$

is negative.

REMARK. In most cases O_g is hyperbolic but few exceptions which can be easily studied [DH]. If g has at least three fixed branched points, then it will have hyperbolic orbifold. For Newton maps we consider in this paper it is always the case. In general, $\#P_g \geq 5$ suffices to make the orbifold of g hyperbolic.

Now we are ready to state Thurston's theorem for marked branched coverings as given in [PT, Theorem 3.1] and proved in [DH].

Theorem 5.6 (Marked Thurston Theorem). *Let (g, X) be a marked branched covering with hyperbolic orbifold. It is Thurston equivalent to a marked rational map (f, Y) if and only if it has no irreducible Thurston obstruction, i.e. if $\lambda(\Gamma) < 1$ for each irreducible multicurve Γ . In this case, the rational map f is unique up to automorphism of $\widehat{\mathbb{C}}$. \square*

5.2. Arcs Intersecting Obstructions. We present a theorem of Kevin Pilgrim and Tan Lei that is useful to show that certain marked branched coverings are equivalent to rational maps. Again, we first need to introduce some notation.

Let (g, X) be a marked branched covering of degree $d \geq 3$.

Definition 5.7 (Arc System). *An arc in (\mathbb{S}^2, X) is a map $\alpha : [0, 1] \rightarrow \mathbb{S}^2$ such that $\alpha(\{0, 1\}) \subset X$, $\alpha((0, 1)) \cap X = \emptyset$, α is a continuous mapping, injective on $(0, 1)$. The notion of isotopy relative X extends to arcs and is also denoted by \simeq .*

A set of pairwise non-isotopic arcs in (\mathbb{S}^2, X) is called an arc system. Two arc systems Λ, Λ' are isotopic if each curve in Λ is isotopic relative X to a unique element of Λ' and vice versa.

Note that arcs connect marked points (the endpoints of an arc need not be distinct) while simple closed curves run around them. We will see that this leads to intersection properties that will give us some control over the location of possible Thurston obstructions. Since arcs and curves are only defined up to isotopy, we make precise what we mean by arcs and curves intersecting.

Definition 5.8 (Intersection Number). *Let α and β each be an arc or a simple closed curve in (\mathbb{S}^2, X) . Their intersection number is*

$$\alpha \cdot \beta := \min_{\alpha' \simeq \alpha, \beta' \simeq \beta} \# \{(\alpha' \cap \beta') \setminus X\}.$$

The intersection number extends bilinearly to arc systems and multicurves.

If λ is an arc in (\mathbb{S}^2, X) , then the closure of a component of $g^{-1}(\lambda \setminus X)$ is called a *lift* of λ . Each arc clearly has d distinct lifts. If Λ is an arc system, an arc system $\tilde{\Lambda}$ is called a *lift* of Λ if each $\tilde{\lambda} \in \tilde{\Lambda}$ is a lift of some $\lambda \in \Lambda$.

If Λ is an arc system, we introduce a linear map g_Λ on the real vector space \mathbb{R}^Λ similar as for multicurves: for $\lambda \in \Lambda$, set

$$g_\Lambda(\lambda) := \sum_{\lambda' \subset g^{-1}(\lambda)} [\lambda'],$$

where $[\lambda']$ denotes the isotopy class of λ' relative X . Again, the sum is taken to be zero if λ has no preimages in the isotopy class of λ' . We say that Λ is *irreducible* if the matrix representing g_Λ is.

Denote by $\tilde{\Lambda}(g^{on})$ the union of those components of $g^{-n}(\Lambda)$ that are isotopic to elements of Λ relative X , and define $\tilde{\Gamma}(g^{on})$ in an analogous way. Note that if Λ is irreducible, each element of Λ is isotopic to an element of $\tilde{\Lambda}(g^{on})$.

The following theorem is Theorem 3.2 of [PT]. It shows that up to isotopy, irreducible Thurston obstructions cannot intersect the preimages of irreducible arc systems (except possibly the arc systems themselves). We will use this theorem to show that the extended map of an abstract Newton graph is Thurston equivalent to a rational map.

Theorem 5.9 (Arcs Intersecting Obstructions). *Let (g, X) be a marked branched covering, Γ an irreducible Thurston obstruction and Λ an irreducible arc system. Suppose furthermore that $\#(\Gamma \cap \Lambda) = \Gamma \cdot \Lambda$. Then, exactly one of the following is true:*

- (1) $\Gamma \cdot \Lambda = 0$ and $\Gamma \cdot g^{-n}(\Lambda) = 0$ for all $n \geq 1$.
- (2) $\Gamma \cdot \Lambda \neq 0$ and for $n \geq 1$, each component of Γ is isotopic to a unique component of $\tilde{\Gamma}(g^{on})$. The mapping $g^{on} : \tilde{\Gamma}(g^{on}) \rightarrow \Gamma$ is a homeomorphism and $\tilde{\Gamma}(g^{on}) \cap (g^{-n}(\Lambda) \setminus \tilde{\Lambda}(g^{on})) = \emptyset$. The same is true when interchanging the roles of Γ and Λ . \square

5.3. The Realization of Abstract Newton Graphs. We conclude with a proof of Theorem 1.6. If $\hat{\Delta}$ is the channel diagram of the postcritically fixed Newton map f , recall that $\hat{\Delta}_n$ denotes the component of $f^{-n}(\hat{\Delta})$ that contains $\hat{\Delta}$. By $\hat{\Delta}'_n$ we denote the set of vertices of $\hat{\Delta}_n$.

Proof of Theorem 1.6. Let $\Delta \subsetneq \Gamma$ be the abstract channel diagram of Γ . First observe that by condition (2) of Definition 4.6, the vertices v_1, \dots, v_{d_Γ} of Δ are branch points of \bar{g} . Since $d_\Gamma \geq 3$, \bar{g} has hyperbolic orbifold and it suffices to show that (\bar{g}, X) has no irreducible Thurston obstruction: it then follows from Theorem 5.6 that \bar{g} is Thurston equivalent to a rational map f of degree d_Γ , which is unique up to Möbius transformation. Then, f has $d_\Gamma + 1$ fixed points, d_Γ of which are superattracting because \bar{g} has marked fixed branch points v_1, \dots, v_{d_Γ} . The last fixed point is repelling [Mi2, Corollary 12.7 & 14.5] and after possibly conjugating f with a Möbius transformation, we may assume that it is at ∞ . Now it follows from Definition 1.1 that f is a Newton map. It is unique up to a Möbius transformation fixing ∞ , hence up to affine conjugacy.

So suppose by way of contradiction that Π is an irreducible Thurston obstruction for (\bar{g}, X) and let $\gamma \in \Pi$. Then, γ is a non-peripheral simple closed curve in $\mathbb{S}^2 \setminus X$. It is easy to see that each edge λ of Δ forms an irreducible arc system and $\Pi \cdot \lambda = \#(\Pi \cap \lambda)$, hence Theorem 5.9 implies that $\gamma \cdot (\bar{g}^{-n}(\lambda) \setminus \lambda) = 0$ for all $n \geq 1$. Since this is true for all edges of Δ and all vertices of Γ are marked points, we get that $\gamma \cdot (\overline{\Gamma \setminus \Delta}) = 0$. But since $\overline{\Gamma \setminus \Delta}$ is connected and contains $X \setminus \{v_0\}$, this means that γ is peripheral, a contradiction.

In order to prove the last claim, note that (\bar{g}_1, Γ'_1) and (\bar{g}_2, Γ'_2) are Thurston equivalent as marked branched coverings. Let $h : (\mathbb{S}^2, \Gamma'_1) \rightarrow (\mathbb{S}^2, \Gamma'_2)$ be a homeomorphism that conjugates \bar{g}_1 to \bar{g}_2 on Γ'_1 . If e is an edge of Γ_1 with endpoints $x_1, x_2 \in \Gamma'_1$, then $h(e)$ connects $h(x_1)$ with $h(x_2)$. Moreover, h preserves the cyclic order at each vertex of Γ_1 , because it is a homeomorphism of \mathbb{S}^2 . So if $h' : h(\Gamma_1) \rightarrow \Gamma_2$ is a homeomorphism that maps each $h(e)$ to the edge of Γ_2 between $h(x_1)$ and $h(x_2)$, then $h' \circ h$ realizes an equivalence between the two abstract Newton graphs. \square

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